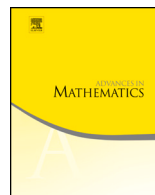




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Corrigendum

Corrigendum to “Evaluation of motivic functions, non-nullity, and integrability in fibers” [Adv. Math. 409 (Part A) (2022) 108635] ☆

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ABSTRACT

We correct the statements and proofs of the (auxiliary) Propositions 4.1 and 4.2 of our paper “Evaluation of motivic functions, non-nullity, and integrability in fibers” in this journal (2022), and we explain how the proofs of the main results can be adapted to work with those corrected propositions.

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Motivic constructible functions
 Grothendieck rings
 Integrability

The authors regret that the proof of the furthermore part of [1, Proposition 4.1] does not work in the presence of torsion. This then also affects one part of [1, Proposition 4.2], where that furthermore part is used. In this corrigendum, we formulate and prove corrected versions of the wrong parts of these two propositions (as Propositions 1.4 and 3.1), and we explain how to adapt the proofs of the main results of [1] to use the corrected statements (as Propositions 1.3 and 3.2). In Sections 1 and 2, we deal with the results related to nullity of motivic functions, namely [1, Proposition 4.1] and its application in the proof of [1, Theorem 1]. Section 3 deals with the statements related to integrability, namely [1, Proposition 4.2] and its application in the proof of [1, Theorem 2].

1. Corrected statements concerning nullity

We use notation and terminology from [1]. In particular, we work in valued fields using an expansion $\mathcal{L}$ of the Denef-Pas language by constant symbols. As in [1], we need to work in a non-elementary class $\mathcal{S}$ of valued fields, all of which are of the form $(k((t)), k, \mathbb{Z})$, for some fields k of characteristic 0, and we emphasize this by writing “ $\mathcal{S}$ -definable”. Recall that non-elementarity of the class $\mathcal{S}$ is relevant for the notion of a “point” of an $\mathcal{S}$ -definable set X : A point is a pair (K, x) with $K \in \mathcal{S}$ and $x \in X_K$. For details about this and the related notation, we refer the reader to [1, Sections 2, 3.1]. However, for the purpose of this corrigendum, one could mostly pretend to just work in one fixed valued field of the form $k((t))$ all the time.

In [1, Sections 3.2, 3.3], several classes of motivic functions are introduced, in line with the classes of motivic functions introduced in [2] and [3]. This entire corrigendum applies to the two classes $\mathcal{C}$ and $\mathcal{C}^e$. We introduce a short hand notation to prove things for both of them simultaneously.

Notation 1.1. Given any $\mathcal{S}$ -definable set X , $\mathcal{C}^\square(X)$ either stands for $\mathcal{C}(X)$ or for $\mathcal{C}^e(X)$. By a “motivic function on X ”, we mean an element of $\mathcal{C}^\square(X)$.

Almost the entire corrigendum is about motivic functions of a specific shape. We fix this once and for all.

Assumption 1.2. Let A be any $\mathcal{S}$ -definable set and let

$$f = \sum_{(a,b) \in L} c_{a,b} \cdot g^a \cdot \mathbb{L}^{b \cdot g} \in \mathcal{C}^\square(\mathrm{VG}_{\geq 0}^r \times A) \quad (1.1)$$

be a motivic function, where $L \subset \mathbb{N}^r \times \mathbb{Z}^r$ is a finite set, $c_{a,b} \in \mathcal{C}^\square(A)$, $g = (g_1, \dots, g_r)$ is the coordinate projection from $\mathrm{VG}_{\geq 0}^r \times A$ to $\mathrm{VG}_{\geq 0}^r$, g^a stands for $g_1^{a_1} \cdots g_r^{a_r}$ and $b \cdot g$ stands for $b_1 \cdot g_1 + \cdots + b_r \cdot g_r$. We sometimes implicitly set $c_{a,b} = 0$ for $(a, b) \notin L$.

Consider the following statements about such f :

- (Eval=0) For every point x on $\mathrm{VG}_{\geq 0}^r$, the restriction of $f|_{\{x\} \times A}$ is zero (as an element of $\mathcal{C}^\square(\{x\} \times A)$).
- (Fct=0) $f = 0$ (as an element of $\mathcal{C}^\square(\mathrm{VG}_{\geq 0}^r \times A)$).
- (Coeff=0) For all $(a, b) \in L$, we have $c_{a,b} = 0$ (in $\mathcal{C}^\square(A)$).

Clearly, we have $(\text{Coeff}=0) \Rightarrow (\text{Fct}=0) \Rightarrow (\text{Eval}=0)$. The “furthermore” part of [1, Proposition 4.1] claims that one also has the reverse implication $(\text{Fct}=0) \Rightarrow (\text{Coeff}=0)$. This is potentially wrong, as Example 1.6 shows. (In the proof, we were not careful enough concerning torsion elements in $\mathcal{C}^\square(\mathrm{VG}_{\geq 0}^r \times A)$.) That furthermore part is used in two places, namely the proofs of [1, Proposition 4.2] and [1, Theorem 1]. For the moment, let us consider the latter one. (Proposition 4.2 is related to integrability; we will come back to it in Section 3.)

Theorem 1 of [1] states, among others, that if a motivic function $f \in \mathcal{C}^\square(X)$ evaluates to 0 at every point on X , then f itself is 0. Case 2 on [1, p. 23] proves Theorem 1 under the assumption that $X \subset \mathrm{VG}^R \times \mathrm{RF}^n$. This is reduced to the special case where $X = \mathrm{VG}_{\geq 0}^r \times A$, for some $A \subset \mathrm{RF}^N$, and where f is as in Assumption 1.2. Using Case 1 on [1, p. 22], one deduces (from the assumption that f evaluates to 0 at every point on X) that f satisfies (Eval=0), so to finish the proof of Case 2, it remains to prove:

Proposition 1.3. *For $A, f, L, \dots$ as in Assumption 1.2, we have $(\text{Eval}=0) \Rightarrow (\text{Fct}=0)$.*

This is where the furthermore part of [1, Proposition 4.1] is used. Inspecting that proof carefully shows not only that it can be adapted to use the modified furthermore part that we state below as Proposition 1.4, but moreover, that the key ingredients of the proof of Proposition 1.4 itself are also already contained there. We will nevertheless give detailed proofs of Propositions 1.3 and 1.4 in this corrigendum.

To state the corrected furthermore part, we introduce weaker version of the above statements about f , namely, where “ $= 0$ ” has been replaced by “is torsion”:

- (EvalTor) For every point x on $\mathrm{VG}_{\geq 0}^r$, the restriction of f to $\{x\} \times A$ is a torsion element of $\mathcal{C}^\square(\{x\} \times A)$ (meaning that there exists a non-zero integer N such that $N \cdot f(x) = 0$).
- (FctTor) f is a torsion element of $\mathcal{C}^\square(\mathrm{VG}_{\geq 0}^r \times A)$.
- (CoeffTor) For all $(a, b) \in L$, $c_{a,b}$ is a torsion element of $\mathcal{C}^\square(A)$.

We shall prove the following two variants of the furthermore part of [1, Proposition 4.1]:

Proposition 1.4. *For $A, f, L, \dots$ as in Assumption 1.2, we have the following:*

- (1) *If $L \subset \{0\} \times \mathbb{Z}^r$, then $(\text{Eval}=0) \Leftrightarrow (\text{Fct}=0) \Leftrightarrow (\text{Coeff}=0)$.*
- (2) *$(\text{EvalTor}) \Leftrightarrow (\text{FctTor}) \Leftrightarrow (\text{CoeffTor})$.*

So: The original version $(\text{Fct}=0) \Rightarrow (\text{Coeff}=0)$ of the furthermore part holds under an additional assumption on L ; without assumption on L , we at least have the torsion variant $(\text{FctTor}) \Rightarrow (\text{CoeffTor})$. Moreover, we already moved large parts of the proof of Proposition 1.3 into Proposition 1.4 by proving the stronger conclusions $(\text{Eval}=0) \Rightarrow (\text{Coeff}=0)$ and $(\text{EvalTor}) \Rightarrow (\text{CoeffTor})$.

The proof of Proposition 1.3 needs one more ingredient:

Lemma 1.5. *For $A, f, L, \dots$ as in Assumption 1.2, if $(\text{Eval}=0)$ and (CoeffTor) hold, then for each b , we have $\sum_a c_{a,b} g^a = 0$. In particular, $f = 0$.*

Proof of Proposition 1.3. First note that $(\text{Eval}=0)$ trivially implies (EvalTor) ; then apply Proposition 1.4(2) to get (CoeffTor) , and finally use the lemma to deduce $(\text{Fct}=0)$. $\square$

Before proceeding with the main proofs, we give an example showing that the implication $(\text{Fct}=0) \Rightarrow (\text{Coeff}=0)$ is most likely really wrong in general (namely, as soon as $\mathcal{C}(A)$ contains torsion elements).

Example 1.6. Suppose $\mathcal{C}(A)$ contains a non-zero element c which is 2-torsion, that is, $2c = 0$, for some $\mathcal{S}$ -definable set A , and consider the function $f \in \mathcal{C}(\text{VG}_{\geq 0} \times A)$ given by $f = cg + cg^2$ (where as usual, $g: \text{VG}_{\geq 0} \times A \rightarrow \text{VG}_{\geq 0}$ is the coordinate projection). Clearly, f does not satisfy $(\text{Coeff}=0)$, but one easily verifies that f satisfies $(\text{Eval}=0)$, and hence also $(\text{Fct}=0)$ (e.g. by Lemma 1.5).

In the next section, we prove Proposition 1.4 and Lemma 1.5. As already announced, the other place where the furthermore part of [1, Proposition 4.1] is used (namely [1, Proposition 4.2]) will be treated in Section 3.

2. Proofs concerning nullity

We start by giving a proof of Proposition 1.4(2). We will not give a separate proof of Proposition 1.4(1), since it has exactly the same structure; one just needs to replace “is torsion” by “is zero” in the obvious places and drop the part of the proof where L is not a subset of $\{0\} \times \mathbb{Z}^r$. (In that part, torsion cannot be avoided.)

Proof of Proposition 1.4(2). It suffices to prove $(\text{EvalTor}) \Rightarrow (\text{CoeffTor})$, since we trivially have $(\text{CoeffTor}) \Rightarrow (\text{FctTor}) \Rightarrow (\text{EvalTor})$.

It is enough to prove the $r = 1$ case. Indeed, the general case can be reduced to that case using induction as follows: Set $A' = \text{VG}_{\geq 0}^{r-1} \times A$ and write f as a sum of the form (1.1) “relative to A' ”, i.e.,

$$f = \sum_{(a_1, b_1) \in L'} c'_{a_1, b_1} \cdot g^{a_1} \cdot \mathbb{L}^{b_1 \cdot g_1}, \quad (2.2)$$

for some suitable $L' \subset \mathbb{N} \times \mathbb{Z}$ and some $c'_{a_1, b_1} \in \mathcal{C}^\square(A')$. More precisely, we have

$$c'_{a_1, b_1} = \sum_{(a_2, \dots, a_n, b_2, \dots, b_n)} c_{(a_1, \dots, a_n), (b_1, \dots, b_n)} \cdot g_2^{a_2} \cdots g_n^{a_n} \cdot \mathbb{L}^{b_2 \cdot g_2 + \cdots + b_n \cdot g_n}. \quad (2.3)$$

By assumption, $f|_{\{x\} \times A}$ is torsion for point x on $\text{VG}_{\geq 0}^r$. Fixing a point x_1 on $\text{VG}_{\geq 0}$ and applying the inductive $(\text{EvalTor}) \Rightarrow (\text{FctTor})$ to the restriction $f|_{\{x_1\} \times \text{VG}_{\geq 0}^{r-1} \times A}$ yields that $f|_{\{x_1\} \times \text{VG}_{\geq 0}^{r-1} \times A}$ is torsion. Doing this for all x_1 shows that f satisfies (EvalTor) relative to $A' = \text{VG}_{\geq 0}^{r-1} \times A$. We now apply the $r = 1$ case of $(\text{EvalTor}) \Rightarrow (\text{CoeffTor})$ to the sum (2.2) to get that c'_{a_1, b_1} is torsion for all $(a_1, b_1) \in L'$. Finally, we apply the inductive $(\text{FctTor}) \Rightarrow (\text{CoeffTor})$ to each c'_{a_1, b_1} , which yields that $c_{a, b}$ is torsion, as desired.

So from now on, we assume $r = 1$. We may additionally assume that $L \subset \mathbb{N} \times \mathbb{N}$; otherwise, multiply f by a suitable power of $\mathbb{L}^g$, which is invertible in $\mathcal{C}^\square(\text{VG}_{\geq 0}^r \times A)$. Clearly, we may also assume that $L \neq \emptyset$.

We do an induction on $(a_{\max}, b_{\max}) := \max L$, with respect to the lexicographic order.¹

In the case $(a_{\max}, b_{\max}) = (0, 0)$, the only coefficient is $c_{0,0}$, and we essentially have $c_{0,0} = f|_{\{0\} \times A}$ (where “essentially” means: up to identifying A with $\{0\} \times A$), so we trivially have $(\text{EvalTor}) \Rightarrow (\text{CoeffTor})$.

Suppose next that $a_{\max} = 0$ and $b_{\max} \geq 1$. We consider $f_{\text{new}} \in \mathcal{C}^\square(\text{VG}_{\geq 0} \times A)$, which, in slightly sloppy notation, is defined by $f_{\text{new}}(x, w) := f(x+1, w) - \mathbb{L}^{b_{\max}} f(x, w)$. Writing out the sum defining f yields

$$f_{\text{new}} = \sum_{(0, b) \in L} c_{0, b} (\mathbb{L}^b - \mathbb{L}^{b_{\max}}) \mathbb{L}^{bg}.$$

Since the $(0, b_{\max})$ summand cancels, we can apply induction to f_{new} (note that (EvalTor) for f implies (EvalTor) for f_{new}) and obtain that for each $b < b_{\max}$, the coefficient $c_{0, b} (\mathbb{L}^b - \mathbb{L}^{b_{\max}})$ of f_{new} is torsion. Since $\mathbb{L}^b - \mathbb{L}^{b_{\max}}$ is invertible in $\mathcal{C}^\square(A)$, we obtain that $c_{0, b}$ is torsion for all $b < b_{\max}$. It remains to show that $c_{0, b_{\max}}$ is torsion. To this

¹ Note that the order of induction is different than in [1, p. 23]. This is not essential, but makes the proof cleaner.

end, consider $f|_{\{0\} \times A} = \sum_b c_{0,b}$. Since $f|_{\{0\} \times A}$ is torsion by assumption, and since all but one summand of the sum are also torsion, so is the last remaining summand $c_{0,b_{\max}}$.

Finally, suppose that $a_{\max} > 0$. (This part is irrelevant for the proof of Proposition 1.4(1).) We again consider $f_{\text{new}}(x, w) := f(x + 1, w) - \mathbb{L}^{b_{\max}} f(x, w)$, and denote its coefficients by $c'_{a,b}$, i.e., $f_{\text{new}} = \sum_{a,b} c'_{a,b} g^a \mathbb{L}^{bg}$. Again, (EvalTor) holds for f_{new} , and since $c'_{a_{\max}b_{\max}} = 0$, we can apply induction and hence obtain that all $c'_{a,b}$ are torsion. We have

$$\begin{aligned} c'_{a_{\max}-1, b_{\max}} &= \underbrace{a_{\max} \cdot c_{a_{\max}, b_{\max}} \mathbb{L}^{b_{\max}} + c_{a_{\max}-1, b_{\max}} \mathbb{L}^{b_{\max}}}_{\text{contribution from } f(x+1, w)} - \underbrace{c_{a_{\max}-1, b_{\max}} \mathbb{L}^{b_{\max}}}_{\text{contribution from } \mathbb{L}^{b_{\max}} f(x, w)} \\ &= a_{\max} \cdot c_{a_{\max}, b_{\max}} \mathbb{L}^{b_{\max}}, \end{aligned}$$

so from $c'_{a_{\max}-1, b_{\max}}$ being torsion, we obtain that $c_{a_{\max}, b_{\max}}$ is torsion. (Here, we use that $\mathbb{L}^{b_{\max}}$ is invertible.) To get that the remaining coefficients of f are also torsion, apply induction once more, this time to $f - c_{a_{\max}, b_{\max}} g^{a_{\max}} \mathbb{L}^{b_{\max}g}$. $\square$

Proof of Lemma 1.5. Set $f_b := \sum_a c_{a,b} g^a$ (so that $f = \sum_b f_b \mathbb{L}^{b \cdot g}$). Recall that we assume (Eval=0) and (CoeffTor) and want to prove that $f_b = 0$.

Fix a positive integer N such that $N \cdot c_{a,b} = 0$ for all $(a, b) \in L$. For each $i = (i_1, \dots, i_r) \in \{0, \dots, N-1\}^r$, let s_i be the $\mathcal{S}$ -definable map $\text{VG}_{\geq 0}^r \times A \rightarrow \text{VG}_{\geq 0}^r \times A$ sending $(x, w) \in \mathbb{N}^r$ to $(Nx + i, w)$, and let $f_i := s_i^*(f)$ and $f_{i,b} := s_i^*(f_b)$ be the pullbacks. Then (Eval=0) for f clearly implies (Eval=0) for f_i , i.e., $f_i|_{\{x\} \times A} = 0$ for every point x on $\text{VG}_{\geq 0}^r$. Note that

$$f_i = \sum_b f_{i,b} \cdot \mathbb{L}^{b \cdot (Ng + i)} \quad (2.4)$$

and

$$f_{i,b} = \sum_a c_{a,b} \cdot (Ng + i)^a$$

(where we use some sloppy notation: $Ng + i$ is the map sending a point (x, w) on $\text{VG}_{\geq 0}^r \times A$ to the point $Nx + i$ of $\text{VG}_{\geq 0}^r$). After multiplying out the powers $(Ng + i)^a$ and dropping all those summands which have a factor N (they disappear due to the factor $c_{a,b}$), one obtains

$$f_{i,b} = \sum_a c_{a,b} \cdot i^a,$$

which does not depend on g anymore. (More formally, $f_{i,b}$ is the pullback of a motivic function on A along the projection $\text{VG}_{\geq 0}^r \times A \rightarrow A$.) Thus Proposition 1.4(1) applies to (2.4) and yields, using (Eval=0) $\Rightarrow$ (Fct=0), that $f_{i,b} = 0$. Since this holds for all i , this implies $f_b = 0$, as claimed. $\square$

In the next section, we will need a lemma whose proof is quite similar to the above one. We prove it right away, while the reader (hopefully) still has the above proof in mind:

Lemma 2.1. *Suppose that all structures in $\mathcal{S}$ are elementarily equivalent (so that we can identify the set of points on $\mathrm{VG}_{\geq 0}$ with $\mathbb{N}$) and that $f = \sum_a c_a g^a$ is a motivic function on $\mathrm{VG}_{\geq 0}$ satisfying (CoeffTor) (i.e., each c_a is torsion). Suppose moreover that (Eval=0) holds almost everywhere, in the sense that the restriction $f|_{\{x\} \times A}$ is zero for all but finitely many $x \in \mathbb{N}$. Then $f = 0$.*

Proof. We start as in the proof of Lemma 1.5: Fix $N > 0$ such that $N \cdot c_a = 0$ for all a , define $s_i: \mathrm{VG}_{\geq 0} \rightarrow \mathrm{VG}_{\geq 0}, x \mapsto Nx + i$ as before for $i \in \{0, \dots, N-1\}$ and let $f_i := s_i^*(f)$ be the pullback. The computation we did for $f_{i,b}$ in the above proof now applies directly to f_i and yields $f_i = \sum_a c_a \cdot i^a$. Since (Eval=0) holds for f almost everywhere, it in particular holds at some points in the image of s_i , so there exists at least one $x \in \mathbb{N}$ such that $f_i|_{\{x\} \times A} = 0$. Since f_i does not depend on x , this implies $f_i = 0$. Since this holds for all i , we get $f = 0$. $\square$

3. Integrability

The second place where the furthermore part of [1, Proposition 4.1] is used (namely, apart from [1, Theorem 1]) is [1, Proposition 4.2], which in turn is used in the proof of [1, Theorem 2]. That Proposition 4.2 lists some statements which are claimed to be equivalent to some motivic function being integrable in relative dimension 0. Statement (iii) from the proposition however is too strong to be equivalent. Proposition 3.1 below is a corrected version of the equivalence between integrability and (iii), but for simplicity, this corrected version is stated in a less general setting than [1, Proposition 4.2]. That less general version is enough for the application in the proof of [1, Theorem 2]. The notion of integrability for motivic functions is recalled in [1, Section 3.4], following [2] and [3].

Proposition 3.1. *Let $A, f, L, \dots$ be as in Assumption 1.2, and suppose moreover that A is of the form $A = \mathbb{R}F^n \times Z$, for some $\mathcal{S}$ -definable set Z . Set $L_0 := L \cap (\mathbb{N}^r \times (Z \setminus \mathbb{N})^r)$. Then the following are equivalent:*

- (1) *f is integrable in relative dimension 0 over Z , along the projection $X \rightarrow Z$.*
- (2) *The sub-sum $f_\infty := \sum_{(a,b) \in L \setminus L_0} c_{a,b} \cdot g^a \cdot \mathbb{L}^{b \cdot g}$ is equal to 0 (as an element of $\mathcal{C}^\square(\mathrm{VG}_{\geq 0}^r \times A)$).*

The stronger statement [1, Proposition 4.2](iii) claimed, instead of $f_\infty = 0$, that $c_{a,b} = 0$ for all $(a,b) \in L \setminus L_0$. Our above Example 1.6 is also a potential counter-

example to that: The function f in that example is integrable since it is zero, but $c_{1,0}$ and $c_{2,0}$ are non-zero.

Remark 3.1. In the more general setting of [1, Proposition 4.2], f itself is not required to be of the shape from Assumption 1.2. However, one can bring any motivic function f into that shape by partitioning the domain into finitely many pieces X_i and pulling back along certain affine linear maps θ_i . The strong version of the corrected [1, Proposition 4.2] would be: For every choice of such X_i and θ_i , the original f satisfies Proposition 3.1(1) if and only if each pullback $\theta_i^*(f)$ satisfies Proposition 3.1(2). That full version easily follows from the version given in Proposition 3.1.

Before proving Proposition 3.1, we explain how it can replace the wrong part of [1, Proposition 4.2] in the proof of [1, Theorem 2]. Note that we can freely use [1, Theorem 1] and its corollaries. The place where that part of [1, Proposition 4.2] is used is in Case 1 on [1, p. 25]. There, after some reduction, we are in the situation where we need to prove the following:

Proposition 3.2. *Let $A, f, L, \dots$ be as in Assumption 1.2. Suppose moreover that $A = \mathbb{R}\mathbb{F}^n \times Z$ and that $f|_{\mathrm{VG}_{\geq 0}^r \times \mathbb{R}\mathbb{F}^n \times \{z\}}$ is integrable for every point z on Z . Then f is integrable in relative dimension 0 over Z (along the projection $\mathrm{VG}_{\geq 0}^r \times \mathbb{R}\mathbb{F}^n \times Z \rightarrow Z$).*

Here is a proof of this using Proposition 3.1 instead of [1, Proposition 4.2].

Proof of Proposition 3.2. Applying Proposition 3.1 to $f|_{\mathrm{VG}_{\geq 0}^r \times \mathbb{R}\mathbb{F}^n \times \{z\}}$ yields that the sub-sum

$$\sum_{(a,b) \in L \setminus L_0} c_{a,b} | \mathbb{R}\mathbb{F}^n \times \{z\} \cdot g^a \cdot \mathbb{L}^{b \cdot g}$$

(for L_0 as in Proposition 3.1) is equal to 0. Since this holds for every point z on Z , [1, Corollary 3.6.5] yields that also

$$\sum_{(a,b) \in L \setminus L_0} c_{a,b} \cdot g^a \cdot \mathbb{L}^{b \cdot g}$$

is equal to 0. Thus we have $f = \sum_{(a,b) \in L_0} c_{a,b} g^a \mathbb{L}^{b \cdot g}$, and this is clearly integrable. $\square$

In the proof of Proposition 3.1, we will need the following lemma:

Lemma 3.3. *Let $f = \sum_a c_a g^a$ be a motivic function on $\mathrm{VG}_{\geq 0} \times A$, and denote by $h: \mathrm{VG}_{\geq 0} \times A \rightarrow A$ the projection. Suppose that there exists an $\mathcal{S}$ -definable set $X \subset \mathrm{VG}_{\geq 0} \times A$ such that for each point w on A , the fiber complement $h^{-1}(w) \setminus X$ is finite and such that $f|_X = 0$. Then $f = 0$.*

Proof. By [1, Corollary 3.6.5], it suffices to prove that $f|_{h^{-1}(w)} = 0$ for every point w on A . Thus it suffices to prove the lemma for A replaced by w and $\mathcal{S}$ replaced by $\mathcal{S}(w)$, meaning firstly that A can be entirely dropped from the notation and secondly that all structures in $\mathcal{S}$ are elementarily equivalent, so that we can identify the set of points on $\mathrm{VG}_{\geq 0}$ with $\mathbb{N}$.

Fix any $d \in \mathbb{N}$ such that $i \in X$ for all $i \geq d$, and define $\theta: \mathrm{VG}_{\geq 0} \rightarrow \mathrm{VG}_{\geq 0}$ by $\theta(x) = x + d$. Then the pullback $\theta^*(f) = \sum_a c_a(g+d)^a$ is entirely zero. By Proposition 1.4 (1), $\theta^*(f)$ has torsion coefficients. The coefficient of g^a in $\theta^*(f)$ has the form $c_a + \sum_{a' > a} s_{a'} c_{a'}$ for some integers $s_{a'}$, so using a downwards induction on a , one obtains that all the c_a are also torsion. Now Lemma 2.1 implies that $f = 0$. $\square$

Proof of Proposition 3.1. First note that the implication (2) $\Rightarrow$ (1) is trivial: By (2), f can be written as $f = \sum_{(a,b) \in L_0} c_{a,b} \cdot g^a \cdot \mathbb{L}^{b \cdot g}$, which is integrable essentially by definition. So let us now prove (1) $\Rightarrow$ (2).

We start by treating the case $r = 1$.

Essentially by definition of integrability (and using the correct part of [1, Proposition 4.1]), we find a partition of the domain $\mathrm{VG}_{\geq 0} \times A$ of f into sets X_i and definable bijections $\theta_i: \mathrm{VG}^{r_i} \times A_i \rightarrow X_i$ such that each pullback $f_i := \theta_i^*(f)$ is of the form

$$f_i = \sum_{(a,b) \in L_i} d_{a,b,i} g^a \mathbb{L}^{b \cdot g} \quad (3.5)$$

with $L_i \subset \mathbb{N}^{r_i} \times (\mathbb{Z} \setminus \mathbb{N})^{r_i}$. Moreover, we have $r_i \in \{0, 1\}$ (since we are assuming $r = 1$), and in the case $r_i = 1$, the map θ_i is of the form

$$\theta_i(x, w) = (e_i x + d_i(w), w).$$

for some positive integer e_i and some definable function $d_i: A_i \rightarrow \mathrm{VG}_{\geq 0}$.

Now recall that f was given as a sum $f = \sum_b f_b \mathbb{L}^{b \cdot g}$, where $f_b := \sum_a c_{a,b} g^a$, and that our goal is to prove that $f_\infty = \sum_{b \geq 0} f_b \mathbb{L}^{b \cdot g}$ is zero. We will more precisely prove that $f_b = 0$ for every $b \geq 0$. To this end, it suffices to prove that for each i with $r_i = 1$, the pullback $\theta_i^*(f_b)$ is zero. Indeed, this implies $f_{b|X} = 0$, where X is the union the corresponding images X_i , and that union X satisfies the assumption of Lemma 3.3, which then yields $f_b = 0$.

We now fix an i with $r_i = 1$ for the remainder of the proof of the $r = 1$ case. The pullback of f_b is of the form

$$f_{i,b} = \theta_i^*(f_b) = \sum_a c'_{a,b,i} g^a,$$

and expressing the pullback of f in terms of this gives

$$f_i = \sum_b f_{i,b} \mathbb{L}^{b \cdot (e_i g + d_i)} = \sum_b \mathbb{L}^{b \cdot d_i} f_{i,b} \mathbb{L}^{be_i g}. \quad (3.6)$$

Subtracting (3.5) from (3.6) yields a sum

$$g = \sum_{a,b} d'_{a,b} g^a \mathbb{L}^{be_i}$$

which is zero (as a motivic function). In particular, g satisfies (Eval=0), and by Proposition 1.4(2), its coefficients $d'_{a,b}$ are torsion, so Lemma 1.5 implies that for each fixed b we have

$$\sum_a d'_{a,b} g^a = 0.$$

Using that $d'_{a,be_i} = \mathbb{L}^{bd_i} c'_{a,b,i} - d_{a,be_i,i}$, we obtain

$$\sum_a (\mathbb{L}^{bd_i} c'_{a,b,i} - d_{a,be_i,i}) g^a = 0 \quad (3.7)$$

for every b . Now recall that for $b \geq 0$, we have $d_{a,be_i,i} = 0$, so after multiplying by $\mathbb{L}^{-bd_i}$, (3.7) becomes $f_{i,b} = 0$, which is what we had to show in the case $r = 1$.

Let us now come back to general r . To simplify notation, we set $c_{a,b} = 0$ for all $(a,b) \notin L$, so that we can omit L from the notation. We first prove a variant of the desired result (1) $\Rightarrow$ (2), namely that (1) implies

$$\sum_{(a,b) \in \mathbb{N}^r \times (\mathbb{N} \times \mathbb{Z}^{r-1})} c_{a,b} g^a \mathbb{L}^{bg} = 0. \quad (3.8)$$

To obtain this, set $A' = \text{VG}_{\geq 0}^{r-1} \times A$ and write f as

$$f = \sum_{(a_1,b_1) \in \mathbb{N} \times \mathbb{Z}} c'_{a_1,b_1} \cdot g^{a_1} \cdot \mathbb{L}^{b_1 \cdot g_1}, \quad (3.9)$$

where

$$c'_{a_1,b_1} = \sum_{(a_2, \dots, a_n, b_2, \dots, b_n) \in \mathbb{N}^{n-1} \times \mathbb{Z}^{n-1}} c_{(a_1, \dots, a_n), (b_1, \dots, b_n)} \cdot g_2^{a_2} \cdots g_n^{a_n} \cdot \mathbb{L}^{b_2 \cdot g_2 + \cdots + b_n \cdot g_n}.$$

That f is integrable over Z implies that it is also integrable over A' , so by the $r = 1$ case applied to (3.9), we have

$$\sum_{(a_1,b_1) \in \mathbb{N} \times \mathbb{N}} c'_{a_1,b_1} g^{a_1} \mathbb{L}^{b_1 g_1} = 0.$$

Plugging in the c'_{a_1,b_1} yields (3.8).

Finally, we are in the position to prove $(1) \Rightarrow (2)$ for arbitrary r . To this end, set

$$f_i := \sum_{(a,b) \in M_i} c_{a,b} g^a \mathbb{L}^{bg},$$

where $M_i = \mathbb{N}^r \times (\mathbb{Z} \setminus \mathbb{N})^i \times \mathbb{Z}^{r-i}$, for $i = 0, \dots, r$. Then $f_r = f - f_\infty$, so to obtain $f_\infty = 0$, we can equivalently show that $f_r = f$.

Clearly we have $f_0 = f$. For $i \geq 1$, we have

$$f_{i-1} - f_i = \sum_{(a,b) \in M_{i-1} \setminus M_i} c_{a,b} g^a \mathbb{L}^{bg},$$

and $M_{i-1} \setminus M_i = \{(a, b) \in M_{i-1} \mid b_i \in \mathbb{N}\}$, so “(1) $\Rightarrow$ (3.8)” applied to f_{i-1} (with the first and the i th coordinate swapped) yields that $f_{i-1} - f_i = 0$. Putting everything together gives $f = f_0 = \dots = f_r$. $\square$

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